ANSWERS TO ANALYSIS III, MID-SEMESTERAL EXAMINATION, 2008, B.MATH 2ND YEAR

1 (i) Given that $f : U \to \mathbb{R}$ is differentiable at $x = (a, b) \in U \subseteq \mathbb{R}^2$.
Define $g : \mathbb{R} \to \mathbb{R}^2$ by $g(t) = (a + t, b + \sin t)$ for all $t \in \mathbb{R}$.
Then $g(0) = (a, b) = x$, g is differentiable at 0 and $g'(0) = \begin{bmatrix} g_1'(0) \\ g_2'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Now

$$\begin{aligned}
\lim_{h\to 0} \frac{f(a+h, b+\sin h) - f(a,b)}{h} &= \lim_{h\to 0} \frac{f\circ g(h) - f\circ g(0)}{h} \\
&= (f\circ g)'(0) \\
&= f'(g(0)) \circ g'(0) \text{ by Chain Rule} \\
&= f'(x) \circ g'(0) \\
&= \begin{bmatrix} D_1 f(x) & D_2 f(x) \end{bmatrix} \begin{bmatrix} g_1'(0) \\ g_2'(0) \end{bmatrix} \\
&= \begin{bmatrix} D_1 f(x) & D_2 f(x) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= D_1 f(x) + D_2 f(x)
\end{aligned}$$

where $D_i f(x)$ is the i th partial derivative of f at $x = (a, b)$

1 (ii) Let $n \geq 2$ and $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a C^1 map satisfying $||f(x)|| = 1$ for all $x \in U$.
We shall prove that the Jacobian Matrix $Df(x)$ of f at any point x is not invertible.
Given that $||f(x)||^2 = f_1(x)^2 + f_2(x)^2 + ... + f_n(x)^2 = 1$.
$\Rightarrow \frac{\partial(f_1(x)^2 + f_2(x)^2 + ... + f_n(x)^2)}{\partial x_i} = 0$ for all $i = 1, 2, ..., n$.
$\Rightarrow 2f_1(x)\frac{\partial f_1(x)}{\partial x_i} + 2f_2(x)\frac{\partial f_2(x)}{\partial x_i} + ... + 2f_n(x)\frac{\partial f_n(x)}{\partial x_i} = 0$ for all $i = 1, 2, ..., n$.

$$\Rightarrow \begin{bmatrix} f_1(x) & f_2(x) & ... & f_n(x) \end{bmatrix} \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_i} \\ \frac{\partial f_2(x)}{\partial x_i} \\ \vdots \\ \frac{\partial f_n(x)}{\partial x_i} \end{bmatrix} = 0 \text{ for all } i = 1, 2, ..., n.$$

$$\Rightarrow \begin{bmatrix} f_1(x) & f_2(x) & ... & f_n(x) \end{bmatrix} \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_i} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_i} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$$

That is $\begin{bmatrix} f_1(x) & f_2(x) & ... & f_n(x) \end{bmatrix} Df(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$

$$\Rightarrow Df(x)^T \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ for all } x \in \mathbb{R}^n$$

Note that $||f(x)|| = 1$ implies that $f(x)$ is a non-zero vector for all $x \in \mathbb{R}^n$. Therefore $Df(x)^T f(x) = 0$ implies that $Df(x)$ is not invertible for all $x \in \mathbb{R}^n$.

2 (i) Let $n \geq 2$. Define $g : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g(y_1, y_2, ..., y_{n-1}) = f(0, y_1, y_2, ..., y_{n-1}) \text{ for all } (y_1, y_2, ..., y_{n-1}) \in \mathbb{R}^{n-1}$$

Fix $(a_2, a_3, ..., a_n) \in \mathbb{R}^{n-1}$ and define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = f(x, a_2, a_3, ..., a_n) \text{ for all } x \in \mathbb{R}.$$

Then $h'(x) = \frac{\partial f}{\partial x_1}(x, a_2, a_3, ..., a_n)$ for all $x \in \mathbb{R}$, where $\frac{\partial f}{\partial x_i}$ is the i th partial derivative of f. It is given that $h'(x) \equiv 0$. Therefore by Fundamental Theorem of Calculus, for all $b > a$ we have

$$h(b) - h(a) = \int_a^b h'(x)dx = 0$$

Thus $f(a, a_2, a_3, ..., a_n) = h(a) = h(b) = f(b, a_2, a_3, ..., a_n)$ for all $a, b \in \mathbb{R}$. This is true for all fixed $(a_2, a_3, ..., a_n) \in \mathbb{R}^{n-1}$. This shows that

$$g(x_2, x_3, ..., x_n) = f(x_1, x_2, ..., x_n) \text{ for all } (x_1, x_2, ..., x_n) \in \mathbb{R}^n.$$

Now note that $\frac{\partial g}{\partial y_i} = \frac{\partial f}{\partial x_{i+1}}$, where $\frac{\partial g}{\partial y_1}$ is the i th partial derivative of g. Hence clearly g is C^1 as f is.

2 (ii) Let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function. Then f is continuous on $\mathbb{R}^n$. Assume that f is injective on $\mathbb{R}^n$.
Then since f is injective and continuous $f(\mathbb{R}^n)$ is a non-trivial interval in $\mathbb{R}$. Let c be an interior point in $f(\mathbb{R}^n)$. Then there exists a unique $y \in \mathbb{R}^n$ such that $f(y) = c$. Now note that $\mathbb{R}^n \setminus \{y\}$ is connected and f is continuous on $\mathbb{R}^n \setminus \{y\}$. But $f(\mathbb{R}^n) \setminus \{c\}$ is disconnected in $\mathbb{R}$. This is a contradiction. Hence f can not be injective.

3 (i) Let $f : [a, b] \to \mathbb{R}$ be a bounded monotonically non-decreasing function. Let $x \in [a, b]$. For $h > 0$ let

$$\Omega_h(f, x) = \sup\{f(y) - f(z) : y, z \in [a, b] \cap (x - h, x + h)\}$$

$$o(f, x) = \lim_{h \to 0} \Omega_h(f, x)$$

Clearly $\Omega_h(f, x) \leq \Omega_{h'}(f, x)$ if $h \leq h'$ and hence $o(f, x) \leq \Omega_h(f, x)$ for all $h > 0$. Thus since f is monotonically non-decreasing we can observe the following for $o(f, x)$:

Case 1: $a < x < b$.
We have $o(f, x) \leq f(z) - f(y)$ for any $y, z \in [a, b]$ with $y < x < z$.

Case 2: $x = a$
We have $o(f, x) \leq f(y) - f(x)$ for any y with $a < y \leq b$

Case 3: $x = b$
We have $o(f, x) \leq f(b) - f(y)$ for any y with $a \leq y < b$.

Without loss of generality assume that $x_1 < x_2 < ... < x_n$. Choose a_i's such that $a = a_0 \leq x_1 < a_1 < x_2 < a_2 < x_3 < ... < x_n \leq a_n = b$. Therefore from the above observations $o(f, x_i) \leq f(a_i) - f(a_{i-1})$. Thus

$$\sum_{i=1}^{n} o(f, x_i) \leq \sum_{i=1}^{n} f(a_i) - f(a_{i-1}) = f(b) - f(a)$$

3 (ii) Let $f : [a, b] \to \mathbb{R}$ be a bounded monotonically non-decreasing function. Let $S_n := \{x \in [a, b] : o(f, x) > \frac{1}{n}\}$. Using part (i) we can see that S_n is a finite set. Thus $\{x \in [a, b] : o(f, x) > 0\} = \cup_{\infty}^{n=1} S_n$ is countable. That is, the set of all discontinuous points of f is countable [see Thm 1-10, Spivak]. Thus f is integrable [see Thm 3-8, Spivak].

4 (i) Let A be a closed subset of $\mathbb{R}^n$ and let U be an open subset of $\mathbb{R}^n$ with $A \subset U$.

Let $U_1 = U, U_2 = \mathbb{R}^n \setminus A$, and let $\mathcal{O} = \{U_1, U_2\}$. Then $\mathcal{O}$ forms an open cover for $\mathbb{R}^n$. Now from [Thm 3-11, Spivak] there is a collection Φ of C^∞ functions ϕ defined on $\mathbb{R}^n$ satisfying

(a) For each $x \in \mathbb{R}^n$ we have $0 \leq \phi(x) \leq 1$

(b) For each $x \in \mathbb{R}^n$ there is an open set V containing x such that all but finitely many $\phi \in \Phi$ are) on V

(c) For each $x \in \mathbb{R}^n$ we have $\sum_{\phi \in \Phi} \phi = 1$

(d) For each $\phi \in \Phi$ there is an open set U in $\mathcal{O}$ such that $\phi = 0$ outside of some closed set contained in U

Consider

$$f = \sum_{\substack{\phi \in \Phi, \\ \phi = 0 \text{ on some } K^c \\ K \subset U_1, \text{ closed}}} \phi$$

Then using (b) we can see that f is C^∞ and we have $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in U^c$ from (c) and (d).

4 (ii) Let $A = \{(x, y) \in \mathbb{R}^2 : 1 \leq xy \leq 2, x \leq y \leq 2x\}$

$$\begin{aligned}
\text{Area of } A &= 2\left(\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\frac{1}{x}}^{2x} dydx + \int_{1}^{\sqrt{2}}\int_{x}^{\frac{2}{x}} dydx\right) \\
&= 2\left(\int_{\frac{1}{\sqrt{2}}}^{1}\left(2x - \frac{1}{x}\right)dx + \int_{1}^{\sqrt{2}}\left(\frac{2}{x} - x\right)dx\right) \\
&= 2((x^2 - \log x)|_{\frac{1}{\sqrt{2}}}^{1} + (2\log x - \frac{x^2}{2})|_1^{\sqrt{2}}) \\
&= 2\log\sqrt{2}
\end{aligned}$$

References

[1] Michael Spivak: *Calculus on Manifolds*, Perseus Books (1965)